Mathathon 2019
Round 1

Maths and Physics Club, IIT Bombay

2^{nd} October, 2019

Name:
E-mail:
Freshie/Senior

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1. Find all $f : \mathbb{R} \to \mathbb{R}$ such that for all $x,\ y \in \mathbb{R}$, the following is satisfied:

$$f(xf(x) + f(y)) = (f(x))^2 + y.$$

2. Let $a,\ b,\ c$ be positive real numbers. Prove that

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{b+a} \geq \frac{3}{2}.$$

3. In the equation $x^3 + \cdots x^2 + \cdots x + \cdots = 0$, A replaces one of the three dots by an integer unequal to 0. Then B replaces one of the remaining dots by an integer. Finally, A replaces the last set of dots by an integer. Prove that A can play so that all three roots of the resulting cubic equations are integers.

4. Let sequences $(a_n)_{n \in \mathbb{N} \cup \{0\}}$ and $(b_n)_{n \in \mathbb{N} \cup \{0\}}$ be defined as follows:

$$a_0 = \frac{1}{2}$$

$$b_0 = \frac{1}{3}$$

$$a_n = a_0 a_{n-1} - b_0 b_{n-1} \qquad (\forall n \in \mathbb{N})$$

$$b_n = a_0 b_{n-1} + b_0 a_{n-1} \qquad (\forall n \in \mathbb{N})$$

Evaluate:

$$\text{(i)} \sum_{n=0}^{\infty} a_n$$

$$\text{(ii)} \sum_{n=0}^{\infty} b_n$$

5. Let $f : [0, 1) \to \mathbb{R}$ be a continuous and thrice-differentiable function satisfying:

$$x^2 f^{(3)}(x) + 3x f^{(2)}(x) + f^{(1)}(x) = \frac{1}{x+1}$$

$$f(0) = 0$$

Where $f^{(n)}$ denotes the $n-$th derivative of f. The function $I : \mathbb{N} \longrightarrow \mathbb{R}$ is defined as:

$$I(n) = \int_0^1 \frac{f(x^n)}{x} \mathrm{d}x$$

Evaluate:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{I(k)}{k}$$

The Riemann-Zeta function is defined as follows:

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}$$

You may use that:

$$\zeta(2)=\frac{\pi^2}{6}$$

$$\zeta(4)=\frac{\pi^4}{90}$$

6. Let $f:\mathbb{R}^+\to\mathbb{R}$ be defined as:

$$f(x)=\frac{\log x}{x}$$

Let $g:\mathbb{N}\to\mathbb{R}$ be defined as:

$$g(n)=\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(\frac{f(x)}{n!}\right)\bigg|_{x=1}$$

Evaluate:

$$\lim_{m\to\infty}\sum_{k=2m}^{4m+1}g(k)\qquad (m\in\mathbb{N})$$

7. Let $a,b,c\in\mathbb{N}$ be such that

$$a^2+b^2=c^2 \text{ and } c-b=1$$

Prove that: a^b+b^a is divisible by c.

8. Let ABC be a triangle with incenter I whose incircle is tangent to $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ at D, E, F. Points P lies on $\overline{EF}$ such that $\overline{DP}\perp\overline{EF}$. Ray BP meets $\overline{AC}$ at Y and ray CP meets $\overline{AB}$ at Z. Point Q is selected on the circumcircle of ΔAYZ so that $\overline{AQ}\perp\overline{BC}$.
Prove that P, I, Q are collinear.

9. Let $\frac{1}{a_1-2\iota}$, $\frac{1}{a_2-2\iota}$, $\frac{1}{a_3-2\iota}$, $\frac{1}{a_4-2\iota}$, $\frac{1}{a_5-2\iota}$, $\frac{1}{a_6-2\iota}$, $\frac{1}{a_7-2\iota}$, $\frac{1}{a_8-2\iota}$ be vertices of a regular octagon, where $a_j\in\mathbb{R}$ for $j=1,\ 2,\ \ldots,\ 8$ and $\iota^2=-1$. Then, find the area of the octagon.

10. Find the number of 10 digit numbers formed by using all the digits $0,\ 1,\ 2,\ \ldots,\ 9$ without repetition such that they are divisible by 11111.